Fractional Supersymmetry through Generalized Anyonic algebra

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Abstract

The construction of anyonic operators and algebra is generalized by using quons operators. Therefore, the particular version of fractional supersymmetry is constructed on the two-dimensional lattice by associating two generalized anyons of different kinds. The fractional supersymmetry Hamiltonian operator is obtained on the two-dimensional lattice and the quantum algebra $U_q(sl_2)$ is realized.

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1 Introduction

A grown interest has been devoted, recently, to fractional statistics[1, 2] describing particles with fractional spin[3, 4, 5]. These particles are called anyons and interpolate between bosons and fermions, it has been shown also that anyons are non-local particles defined on the two-dimensional space. Mathematically, the group symmetry associated to the anyonic systems involves some special Lie algebras appearing as a quantum deformation of the usual Lie algebras and Lie groups[6, 7, 8, 9, 10]. Indeed, one can prove that quantum groups[11] are the mathematical objects allowing the description of these particular systems[12, 13]. Many works have been devoted, in the last few years, to the construction and the realization of quantum groups. We note for example the anyonic algebra[14] leading to the obtaining of these quantum algebras. The anyonic operators are introduced on the two-dimensional lattice as a non-local operators seen as a generalization of generators of the Jordan-Wigner transformation[15].

The present work concerns the study on the two-dimensional lattice of N=2 fractional supersymmetry (FSUSY)[16, 17, 18, 19]. We obtain exactly the same algebraic structure of this latter starting from the introduction of the generalized anyonic algebra. Then we introduce some special operators which can be seen as a generalized anyonic operators. We notice that this construction is different from the one used to reproduce the N=2 FSUSY algebra basing on the quonic algebra[19]. This algebraic study allows us to find the Hamiltonian operator describing one N=2 FSUSY system.

This paper is organized as follows : In section 2, as a generalization, we construct new exotic operators in using quonic operators as basic ones. Therefore we realize the generalized anyonic operators and the corresponding algebra. In the third section, the last new operators and algebra will be used to construct N=2 FSUSY on the two-dimensional lattice corresponding to anyonic systems as a certain coupling of two generalized anyonic oscillators of different kinds (γ and δ)[14]. We proceed as in the work [19] where we have constructed the N=2 FSUSY through two different quons. Another result consists on the realization of the quantum algebra $U_q(sl_2)$ using the supercharges already constructed. Finally, section 4 presents some concluding remarks.

2 Generalized anyonic algebra

Let us recall at first the anyonic operators which are seen as non-local two-dimensional operators interpolating between bosonic and fermionic ones. The generalization of these operators allows us to introduce the generalized anyons. We will use the famous angle function $\Theta(x, y)$ appearing in the construction and the description of anyons in the work [6].

We start by giving a brief review on this angle function. One designs by γ_x the cut associated to each point x that we denote by x_γ on the two-dimensional lattice Ω. Denoting by Ω^* the dual lattice of Ω; it is a set of points $x^* = x + 0^*$ with $0^* = (\frac{1}{2}\epsilon, \frac{1}{2}\epsilon)$ the origin of Ω^* and ϵ its spacing which eventually will be sent to zero. In this case γ_x will be on Ω^* from minus infinity to x^* along the x-axis. One considers another type of cuts. We choose the set of cuts δ_x coming from plus infinity to ${}^*x = x - 0^*$ along the x-axis. Consequently, the two types of cuts γ_x and δ_x involve an ordering and opposite ordering

respectively of points x on Ω. This is described by the following proposition

$$x_\delta < y_\delta \Leftrightarrow x_\gamma > y_\gamma \Leftrightarrow x > y \Leftrightarrow \begin{cases} x_2 > y_2, \\ x_1 > y_1, x_2 = y_2 \end{cases} \tag{1}$$

Owing to the equation (1) the angle functions satisfy

$$\begin{aligned}
\Theta_{\gamma_x}(x,y) - \Theta_{\gamma_y}(y,x) &= \begin{cases} \pi,\ x > y \\ -\pi,\ x < y \end{cases} \\
\Theta_{\delta_x}(x,y) - \Theta_{\delta_y}(y,x) &= \begin{cases} -\pi,\ x > y \\ \pi,\ x < y \end{cases} \\
\Theta_{\delta_x}(x,y) - \Theta_{\gamma_x}(x,y) &= \begin{cases} -\pi,\ x > y \\ \pi,\ x < y \end{cases} \\
\Theta_{\delta_x}(x,y) - \Theta_{\gamma_y}(y,x) &= 0, \forall x, y \in \Omega.
\end{aligned} \tag{2}$$

Now let us introduce the operators $K_i(x_\alpha)$ those called the disorder ones. They are expressed by

$$K_i(x_\alpha) = e^{i\nu \sum\limits_{y \neq x} \Theta_{\alpha_x}(x,y)[N_i(y) - \frac{1}{2}]} \tag{3}$$

with $\alpha_x = \gamma_x$ or δ_x and $i = 1, 2, ..., n, n \in \mathbf{N}$. In the equality (3) $N_i(y)$ is nothing but the number operator of quons on the two-dimensional lattice, defined by the generators $a_i^\dagger(x)$ and $a_i(x)$ as follows

$$\begin{aligned}
a_i^\dagger(x) a_i(x) &= [N_i(x)]_{q_i} \\
a_i(x) a_i^\dagger(x) &= [N_i(x) + \mathbf{1}]_{\mathrm{q_i}}
\end{aligned} \tag{4}$$

where $[x]_q = \frac{q^x - 1}{q - 1}$. The operators $a_i^\dagger(x)$ and $a_i(x)$ are respectively the creation and annihilation quonic operators on the two-dimensional lattice Ω, satisfying the relations

$$\begin{aligned}
[a_i(x), a_j^\dagger(y)]_{q_i^{\delta_{ij}}} &= \delta_{ij}\delta(x,y) \\
[a_i(x), a_j(y)]_{q_i^{\delta_{ij}}} &= 0 \quad \forall x, y, \forall i, j \\
[a_i^\dagger(x), a_j^\dagger(y)]_{q_i^{\delta_{ij}}} &= 0 \quad \forall x, y, \forall i, j \\
[N_i(x), a_j(y)] &= -\delta_{ij}\delta(x,y) a_i(x) \\
[N_i(x), a_j^\dagger(y)] &= \delta_{ij}\delta(x,y) a_i^\dagger(x)
\end{aligned} \tag{5}$$

The Dirac function is defined by

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq 0 \end{cases} \tag{6}$$

In the relations (5), one can consider the operators $a_i^\dagger(x)$, $a_i(x)$ and $N_i(x)$; those are generators of the oscillator algebra describing a system of quons. They are seen as a non-local particles and we define them to satisfy the equality

$$(a_i(x))^{d_i} = (a_i^\dagger(x))^{d_i} = 0 \tag{7}$$

where we take the deformation parameter q_i to be a root of unity, $q_i^{d_i} = 1$ then $q_i = e^{i\frac{2\pi}{d_i}}$. One can show that the irreducible representation space (Fock space) of the algebra (relations (5)) is given by the set

$$F_{i_x} = \{|n_{i_x}>, n_{i_x} = 0, 1, ..., d_i - 1\} \tag{8}$$

where the notation i_x means that this Fock space is introduced in each site of the Ω. the actions of the generators $a_i(x)$, $a_i^\dagger(x)$ and $N_i(x)$ are expressed by the following equalities

$$\begin{array}{c} a_i^\dagger(x)|n_{i_x}>=|n_{i_x}+1>, \quad a_i^\dagger(x)|d_i-1>=0 \\ a_i(x)|n_{i_x}>=[n_{i_x}]_{q_i}|n_{i_x}-1>, \quad a_i(x)|0>=0. \end{array} \tag{9}$$

Now we introduce an operators allowing the description of the N=2 FSUSY on the two-dimensional lattice. This realization is seen as an anyonic one, in the sense that the generators will depend on a function which seems to be the angle function used by the authors in the work [14] when they describe the system of anyons. This leads to the definition of the operators

$$A_i(x_\alpha) = K_i(x_\alpha)a_i(x), \tag{10}$$

where $K_i(x_\alpha)$ is the disorder operator introduced in the equation (3).

One proves that these operators obey the following commutation relations

$$\begin{array}{rcl} [A_i(x_\gamma), A_i(y_\gamma)]_{q_ip^{-1}} & = & 0,\, x > y \\ [A_i^\dagger(x_\gamma), A_i^\dagger(y_\gamma)]_{q_ip^{-1}} & = & 0,\, x > y \\ [A_i(x_\gamma), A_i^\dagger(y_\gamma)]_{q_ip} & = & 0,\, x > y \\ [A_i^\dagger(x_\gamma), A_i(y_\gamma)]_{q_ip} & = & 0,\, x > y \\ [A_i(x_\gamma), A_i^\dagger(x_\gamma)]_{q_i} & = & 1 \\ [A_i(x_\gamma), A_j(y_\gamma)] & = & 0,\, \forall i \neq j,\, \forall x, y \in \Omega \\ [A_i^\dagger(x_\gamma), A_j^\dagger(y_\gamma)] & = & 0,\, \forall i \neq j,\, \forall x, y \in \Omega \\ [A_i^\dagger(x_\gamma), A_j(y_\gamma)] & = & 0,\, \forall i \neq j,\, \forall x, y \in \Omega \\ [A_i(x_\gamma), A_j^\dagger(y_\gamma)] & = & 0,\, \forall i \neq j,\, \forall x, y \in \Omega. \end{array} \tag{11}$$

for the anyonic operators of type γ. We point out that the same results can be obtained for the kind δ in replacing p by p^{-1} in (11). It is obvious thus to get the commutation relations between the different kinds of generalized anyonic operators, we have so

$$\begin{array}{rcl} [A_i(x_\delta), A_j(y_\gamma)]_{q_i^{\delta_{ij}}} & = & 0, \forall\, x, y \in \Omega \\ [A_i(x_\delta), A_j^\dagger(y_\gamma)]_{q_i^{\delta_{ij}}} & = & \delta_{ij}\delta(x,y)p^{-[\sum\limits_{z<x} - \sum\limits_{z>x}][N_i(z)-\frac{1}{2}]} \end{array} \tag{12}$$

with $p = e^{i\nu\pi}$, ν is seen as the statistical parameter[1, 2].

By considering the generalized anyonic oscillators $A_i(x_\alpha)$ and $A_i^\dagger(x_\alpha)$, we can demonstrate the following relations

$$(A_i(x_\alpha))^{d_i} = (A_i^\dagger(x_\alpha))^{d_i} = 0 \tag{13}$$

with $\alpha = \gamma, \delta$ and $i = 1, 2, ..., N$. The equation (13) can be seen as a generalization of the hard-core condition found when we have study the generalized statistics in the works [21, 22]. Indeed, for $d_i = 2$ one recover the result allowing us, in this work, to think about some connexion between the generalized statistics and the anyonic ones. Returning to the present paper, the relation (13) can be seen a nilpotency condition leading to the study of quons on the two-dimensional lattice. This equality is obtained starting from the function

$K_i(x_\alpha)$ discussed in the work [14], which is the subject of the following equations

$$\begin{array}{rcl} K_i^\dagger(x_\alpha)a_i(y) &=& e^{i\nu\Theta_{\alpha_x}(x,y)}a_i(y)K_i^\dagger(x_\alpha) \\ K_i^\dagger(x_\alpha)a_i^\dagger(y) &=& e^{-i\nu\Theta_{\alpha_x}(x,y)}a_i^\dagger(y)K_i^\dagger(x_\alpha) \\ K_i(x_\alpha)a_i(y) &=& e^{-i\nu\Theta_{\alpha_x}(x,y)}a_i(y)K_i(x_\alpha) \\ K_i(x_\alpha)a_i^\dagger(y) &=& e^{i\nu\Theta_{\alpha_x}(x,y)}a_i^\dagger(y)K_i(x_\alpha) \\ K_i^\dagger(x_\alpha)K_i(y_\alpha) &=& K_i(y_\alpha)K_i^\dagger(x_\alpha) \end{array} \tag{14}$$

It is proved that, on the above Fock space, these algebraic relations are coherent with the following equalities

$$\begin{array}{rcl} A_i^\dagger(x_\alpha)|n_{i_x}> &=& e^{i\frac{\nu}{2}\sum\limits_{y\neq x}\Theta_{\alpha_x}(x,y)}|n_{i_x}+1> \\ A_i(x_\alpha)|n_{i_x}> &=& [n_{i_x}]_{q_i}e^{-i\frac{\nu}{2}\sum\limits_{y\neq x}\Theta_{\alpha_x}(x,y)}|n_{i_x}-1> \\ A_i^\dagger(x_\alpha)|d_i-1> &=& 0 \\ A_i(x_\alpha)|0> &=& 0. \end{array} \tag{15}$$

We note that the operators $A_i^\dagger(x_\alpha)$ and $A_i(x_\alpha)$ are seen, owing to the relations (14) and (15), as generalized creation and annihilation anyonic operators respectively.

3 N=2 FSUSY on the two-dimensional lattice

The construction of generalized anyonic operators allows us to realize the N=2 FSUSY. This realization involves two different generalized of anyons and is analogous to the one obtained via two different quons in the work [19]. We introduce then the fractional supercharges on Ω as

$$\begin{array}{rcl} Q_+(x) &=& A_1^\dagger(x_\gamma)A_2(x_\delta) \\ Q_-(x) &=& A_1(x_\delta)A_2^\dagger(x_\gamma). \end{array} \tag{16}$$

basing on the equality (13); one can get

$$(Q_\pm(x))^{d_2} = 0, \tag{17}$$

where d_2 is required to be $< d_1$.

In using the construction given in the relation (16), we can get the action of the above fractional supercharges on the Fock space defined by the tensor product

$$F_x = F_{1_x} \otimes F_{2_x} \tag{18}$$

with F_{1_x} and F_{2_x} correspond respectively to the different generalized anyons used to introduce the fractional supercharges.

Now we can get easily the relations

$$\begin{array}{l} Q_+(x)|n_{1_x}> \otimes |n_{2_x}> = p^{\frac{1}{2}}[n_{2_x}]_{q2}|n_{1_x}+1> \otimes |n_{2_x}-1> \\ Q_-(x)|n_{1_x}> \otimes |n_{2_x}> = p^{\frac{1}{2}}[n_{1_x}]_{q1}|n_{1_x}-1> \otimes |n_{2_x}+1> \end{array} \tag{19}$$

Through this realization of the fractional supercharges $Q_\pm(x)$, we can show that these new operators satisfy the following commutation relation in the case of $x > y$ on Ω

$$q_1Q_+(x)Q_-(y) - q_2Q_-(y)Q_+(x) = \delta(x,y)P[q_1[N_1(x)]_{q_1} - q_2[N_2(x)]_{q_2}] \tag{20}$$

where the operator P is written as

$$P = p^{[\sum_{z<x} - \sum_{z>x}][N_1(z)+N_2(z)-1]} \tag{21}$$

One can remark that the equality (20) is not invariant under the hermitian conjugate. This is related to the fact that these generators involve a complex numbers $q_{1,2}$ and p which are different from ± 1. To avoid this difficulty we introduce the hermitian conjugate of the generators $Q_\pm(x)$ and after calculation it is easy to obtain the conjugate equation of (20) as

$$q_1^{-1}Q_-^\dagger(y)Q_+^\dagger(x) - q_2^{-1}Q_+^\dagger(x)Q_-^\dagger(y) = \delta(x,y)P^{-1}[q_1^{-1}[N_1(x)]_{q_1^{-1}} - q_2^{-1}[N_2(x)]_{q_2^{-1}}], \tag{22}$$

We can verify also that

$$(Q_\pm^\dagger(x))^{d_2} = 0. \tag{23}$$

for $d_2 < d_1$.

Another reason to introduce this operation is to construct the Hamiltonian operator corresponding to this system. We can thus express the FSUSY Hamiltonian operator as

$$\begin{gathered} q_1 Q_+(x)Q_-(y) + q_1^{-1}Q_-^\dagger(y)Q_+^\dagger(x) - q_2 Q_-(y)Q_+(x) - q_2^{-1}Q_+^\dagger(x)Q_-^\dagger(y) = \\ \delta(x,y)[Pq_1[N_1(x)]_{q_1} + P^{-1}q_1^{-1}[N_1(x)]_{q_1^{-1}} - Pq_2[N_2(x)]_{q_2} - P^{-1}q_2^{-1}[N_2(x)]_{q_2^{-1}}]. \end{gathered} \tag{24}$$

Using the relation

$$[N_i(x)]_{q_i^{-1}} = q_i^{1-N_i(x)}[N_i(x)]_{q_i}. \tag{25}$$

The RHS of (24) will be rewritten as

$$RHS = \delta(x,y)[P^{-1}q_1^{-N_1(x)} + Pq_1][N_1(x)]_{q_1} - [P^{-1}q_2^{-N_2(x)} + Pq_2][N_2(x)]_{q_2}, \tag{26}$$

The last equation can be interpreted as a Hamiltonian of the generalized anyonic system investigated. So, we can rewrite this Hamiltonian operator as

$$H(x) = \sum_{i,j=1,2} \epsilon_{ij} \frac{\sin[\nu\pi \sum_{z\in\Omega} \aleph(z) + \frac{\pi}{d_i}(2N_i(x)+1)] - \sin[\nu\pi \sum_{z\in\Omega} \aleph(z) + \frac{\pi}{d_i}]}{\sin\frac{\pi}{d_i}}, \tag{27}$$

where $(\epsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\sum_{z\in\Omega} \aleph(z) = (\sum_{z<x} - \sum_{z>x})(N_1(z) + N_2(z) - 1)$. Up to now, we can recapulate our result which consists on the complet description of the N=2 FSUSY basing on a given anyonic realization.

Furthermore a global version of this realization can be readily constructed as follows

$$H = \sum_{x\in\Omega} H(x), \tag{28}$$

where the global supercharges are defined as

$$Q_\pm = \sum_{x\in\Omega} Q_\pm(x), \tag{29}$$

In addition, we can link the N=2 FSUSY obtained on the two-dimensional lattice to the "local" algebra $U_q(sl_2)$ in considering $q_1 = q_2 = q$. In this particular case we define the three local generators as

$$\begin{array}{c} J_{\pm}(x) = P^{-\frac{1}{2}} q^{-\frac{N_2(x)}{2}} Q_{\pm}(x) \\ J_3(x) = \frac{1}{2}(N_1(x) - N_2(x)). \end{array} \tag{30}$$

We can easily check that these local densities of quantum group generators satisfy the following commutation relations

$$\begin{array}{c} [J_+(x), J_-(y)] = \delta(x,y)[2J_3(x)]_q \\ [J_3(x), J_{\pm}(y)] = \pm\delta(x,y)J_{\pm}(x). \end{array} \tag{31}$$

Thus to define the global generators it is sufficient to write

$$\begin{array}{c} J_{\pm} = \sum_{x\in\Omega} J_{\pm}(x) \\ J_3 = \sum_{x\in\Omega} J_3(x) \end{array} \tag{32}$$

and close the $U_q(sl_2)$ algebra as

$$\begin{array}{c} [J_+, J_-] = [2J_3]_q \\ [J_3, J_{\pm}] = \pm J_{\pm}. \end{array} \tag{33}$$

Consequently, we have close the algebra of $U_q(sl_2)$ generated by $J_{\pm}$ and J_3 which are built out of generalized anyonic oscillators.

4 Conclusion

To conclude, we can summarize the lines of this paper in saying that we have constructed a generalized anyonic operators on the two-dimensional lattice in using the q-bosonic operators or called quonic operators, as a generalization of ones defined in the paper [5]. Moreover, we have realized the N=2 FSUSY on the two-dimensional lattice. Where the supercharges are constructed by coupling two different generalized anyonic operators, and the FSUSY Hamiltonian operator of the corresponding system is given. Thus from the N=2 FSUSY realized the well known algebra $U_q(sl_2)$ is derived.

It will be interesting also to rewrite the obtained FSUSY Hamiltonian, in the connexion with the gauge theory, as a form of which we held the term coincidering with the Chern-Simons term. This point is the matter that we are preparing for the next paper.

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References

[1] J. F. Schonfeld, Nucl. Phys. B**185,** 127 (1981).

[2] F. Wilczek. Phys. Rev. Lett. **48,** 1144 (1982); R. B. Laughlin, Science **242,** 525 (1988).

[3] J. M. Leinaas and J. Myrheim, Nuovo Cimento B**37,** 1 (1977); J. M. Leinaas, "Symmetry and Structural Proporties of Condensed Matter", Poznan1992 (World Sci., Singapore, 1993).

[4] A. Lerda, Anyons: Quantum Mechanics of Particles with Fractional Statistics (Springer-Verlag, Berlin, Germany, 1992).

[5] I. J. R. Aitchison and N. E. Mavromatos, Contomp.Phys., **32,** 219 (1991).

[6] T. Brzezinski, I. L. Egusquiza and A. J. Macfarlane, Phys. Lett. B**311,** 202 (1993).

[7] A. J. Macfarlane, J. Phys. A**22,** 4581 (1989).

[8] M. Chaichan, Phys. Lett. B**234,** 72 (1990).

[9] L. C. Biedenharn, J. Phys. A**22,** L873 (1989).

[10] M. A. Vasiliev, Int. J. Mod. Phys. A**6,** 1115 (1991).

[11] A. P. Isaev, Fiz. E'lem. Chastils A.t Yadra, **26,** 1204 (1995).

[12] M. Frau, M. A R-Monteiro and S. Sciuto, J. Phys. A:Math. Gen. **27,** 801 (1994).

[13] L. Frappat, A. Sciarrino, S. Sciuto, P. Sorba, Phys. Lett. B**369,** 313 (1996).

[14] A. Lerda, S. Sciuto, Nucl. Phys. B**401,** 613 (1993).

[15] P. Jordan and E. P. Wigner, Z. Phys., B**47,** 631 (1928).

[16] A. Lerda, S. Sciuto, Nucl. Phys. B**410,** 577 (1993).

[17] M. Rausch de Traubenberg and P. Simon, Preprint; hep-th/9806188.

[18] L. P. Colatto and J. L. Matheus-Valle, Preprint; hep-th/9504101.

[19] M. Daoud and Y. Hassouni, Prog. Theor. Phys., **96,** 1033 (1997).

[20] M. Daoud, J. Douari and Y. Hassouni, Preprint; IC/98/132 and submitted to Jour. Math. Phys.

[21] M. Daoud and Y. Hassouni, Mod. Phys. Lett, A**96,** 1033 (1997).

[22] M. Daoud, J. Douari and Y. Hassouni, Int. Jou. of Theo. Phys., **36,** 3071 (1997).